

# PROOF OF A CONJECTURE OF DAVILA AND KENTER REGARDING A LOWER BOUND FOR THE FORCING NUMBER IN TERMS OF GIRTH AND MINIMUM DEGREE

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**ABSTRACT.** In this note, we study a dynamic coloring of vertices in a simple graph  $G$ . In particular, one may color an initial set of vertices, with all other vertices being non-colored. Then, at each discrete time step, a colored vertex with exactly one non-colored neighbor will force its non-colored neighbor to become colored. The initial set of colored vertices is called a *forcing set* (*zero forcing set*) if by iterating this aforementioned process, all of the vertices in  $G$  become colored. The *forcing number* (*zero forcing number*) of  $G$  is the cardinality of a minimum forcing set in  $G$ , and is denoted by  $F(G)$ . The main contribution of this note is to prove the following conjecture originally posed by Davila and Kenter in [18], and partially resolved in [16, 18, 20, 21]; namely, if  $G$  is a graph with minimum degree  $\delta \geq 2$  and girth  $g \geq 3$ , then  $F(G) \geq \delta + (\delta - 2)(g - 3)$ .

## 1. INTRODUCTION

Dynamic colorings of the vertices in a graph have seen a rise in application and relations to well studied graph theoretic parameters in recent years. By dynamic colorings, we mean a coloring of the vertices in a graph which may propagate (change the color of) vertices that were not initially colored in the graph. Of these dynamic colorings, and of relation to this note, we highlight that forcing sets (zero forcing sets), and the associated graph invariant known as the forcing number (zero forcing number), are of particular interest. Indeed, the forcing number, denoted  $F(G)$ , has been shown to relate to well studied graph invariants such as the domination number, the total domination number, the connected domination number, the path cover number, the chromatic number, the independence number, and the minimum rank, among others; see for example [1, 4, 9, 10, 11, 14, 15, 16, 17, 18, 20, 21]. Moreover, it has been established that the forcing number and its variants lie within the class of  $NP$ -complete decision problems [9, 15, 13]. Thus, it is desirable to find easily computable bounds for  $F(G)$ .

*Graph Terminology.* For the entirety of this note we will restrict ourselves to undirected finite simple graphs. Let  $G = (V, E)$  be a graph. We will denote the order and size of  $G$  by  $n = |V|$  and  $m = |E|$ , respectively. Two vertices  $v, w \in V$  will be called neighbors, or adjacent vertices, whenever  $vw \in E$ . The *open neighborhood* of  $v \in V$  is the set of neighbors

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of  $v$ , denoted by  $N(v) = N_G(v)$ , whereas the *closed neighborhood* of  $v$  is  $N[v] = N_G[v] = N_G(v) \cup \{v\}$ . The *degree* of  $v \in V$  is the cardinality of its open neighborhood, and is denoted by  $d(v) = |N(v)|$ . The maximum and minimum vertex degrees in  $G$  are denoted by  $\Delta(G)$  and  $\delta(G)$ , respectively.

Given a set of vertices  $S \subseteq V$ , the open neighborhood of  $S$  is defined as  $N(S) = N_G(S) = \bigcup_{v \in S} N(v)$ . The closed neighborhood of  $S$  is defined as  $N[S] = N_G[S] = N(S) \cup S$ . The girth of  $G$ , denoted  $g = g(G)$ , is the size of a smallest cycle which is contained in  $G$  as a subgraph.

The *forcing process* on  $G$  is defined as follows: Let  $S \subseteq V$  be a set of initially colored vertices, while all other vertices said to be non-colored. We say that a vertex  $v \in S$  is  $S$ -colored, while a vertex not in  $S$  is said to be  $S$ -non-colored. At each time step, a colored vertex with exactly one non-colored neighbor will *force* its non-colored neighbor to become colored. If  $v \in V$  is such a vertex, then we say that  $v$  is a *forcing vertex*. Further, if  $v$  is a forcing vertex, we say that  $v$  has been *played*. The set  $S$  is said to be a *forcing set*, if by iteratively applying the forcing process all of  $V$  becomes colored. The *forcing number* of  $G$  is the minimum cardinality of a forcing set in  $G$ , denoted  $F(G)$ . If  $S$  is a forcing set of minimum cardinality in  $G$ , we will say that  $S$  is a  $F(G)$ -set.

*Necessary Tools.* The maximum number of edges in a simple graph of order  $n$  and girth at least  $\ell + 1$  is denoted by  $\text{ex}(n; \{C_3, C_4, \dots, C_\ell\})$ , often referred to as *extremal function*. The following theorem about will be essential for the proof of our main result.

**Theorem 1.** [2] *Let  $n \geq 4$  and  $n + 1 \leq v \leq 2n$  be integers. Then*

$$\text{ex}(v; \{C_3, C_4, \dots, C_n\}) = \begin{cases} v & \text{if } n + 1 \leq v \leq \lfloor 3n/2 \rfloor, \\ v + 1 & \text{if } \lfloor 3n/2 \rfloor + 1 \leq v \leq 2n - 1, \\ v + 2 & \text{if } v = 2n. \end{cases}$$

This statement will be used in form of the following corollaries.

**Corollary 1.** *For  $g \geq 11$ ,  $\text{ex}(g - 2, \{C_3, C_4, \dots, C_{g-6}\}) = \begin{cases} g - 1 & \text{for } g \in \{11, 12, 13\}, \\ g - 2 & \text{for } g \geq 14. \end{cases}$*

**Corollary 2.** *For  $g \geq 11$ ,  $\text{ex}(g - 2, \{C_3, C_4, \dots, C_{g-4}\}) = g - 2$ .*

## 2. MAIN RESULT

In this section, we prove a conjecture which was originally posed, and partially resolved by Davila and Kenter in [18], and again partially resolved by Davila and Henning [16], Genter, Penso, Rautenbach, and Souza [20], and Genter and Rautenbach in [21]. We recall the statement of this conjecture with the following.

**Conjecture 1.** [18] *If  $G$  is a graph with girth  $g \geq 3$  and minimum degree  $\delta \geq 2$ , then*

$$(1) \quad F(G) \geq \delta + (\delta - 2)(g - 3).$$

We remark that Conjecture 1 has been resolved whenever  $g \leq 10$ . Next we provide our main result which resolves Conjecture 1 in the affirmative.

**Theorem 2.** *Conjecture 1 is true.*

*Proof.* Before proceeding with our proof, we remark that our technique is influenced heavily by ideas presented in [16]. Furthermore, since the theorem has been proven whenever girth  $g \leq 10$ ; recall [16, 20, 21], we shall assume girth  $g \geq 11$ . Let  $G$  be a graph with minimum degree  $\delta \geq 2$  and girth  $g \geq 11$ . By way of contradiction, suppose  $S \subseteq V$  is a forcing set with cardinality  $|S| \leq \delta + (\delta - 2)(g - 3) - 1$ . Let  $x_1, \dots, x_t$  be a sequence of played vertices which results in all of  $V$  becoming colored starting with  $S$  as an initial set of colored vertices. We remark that  $t$  denotes the  $t$ -th step of the forcing process where all of  $V$  becomes colored at time step  $t$ . Let  $\bar{S} = V \setminus S$ , and so,  $\bar{S}$  is the set of  $S$ -non-colored vertices. From [16] (see Observation 5), we recall that if  $G$  is a graph with minimum degree  $\delta \geq 2$ , and girth  $g \geq 5$ , then  $n \geq g(\delta - 1)$ . Hence, we obtain the chain of inequalities

$$|\bar{S}| = n - |S| \geq g(\delta - 1) - (\delta - 2)(g - 3) - \delta + 1 = g + 2\delta - 5 \geq g - 1.$$

We next set  $X = \{x_1, \dots, x_{g-2}\}$ , and slightly modifying the notation of [16], we set  $S_1 = S \cap N(x_1)$  and

$$\begin{aligned} S_i &= (S \cap N(x_i)) \setminus \left( \bigcup_{j=1}^i N[x_j] \right) && \text{for } i = 2, \dots, g-2, \\ S_X^* &= \bigcup_{i=1}^{g-2} S_i, \\ S_X &= X \cap (S \setminus S_X^*). \end{aligned}$$

As an immediate consequence of these definitions, we obtain the inequality

$$(2) \quad |S| \geq |S_X| + |S_X^*|.$$

Since  $x_1 \in S_X$ , we make note that  $|S_X| \geq 1$ . We next let  $H = (X, E')$ , where  $E' = \{x_i x_j : \text{dist}_G(x_i, x_j) \leq 2\}$ . Let  $m' = |E'|$  be the size of  $H$ . In the proof of their main theorem, it is shown in [16] that the following lemmas hold true.

**Lemma 1** ([16]).  $|S_X^*| \geq (\delta - 1)(g - 2) - m'$ .

**Lemma 2** ([16]). *If  $x \in X \setminus S_X$ , then  $\text{dist}(x, x') = 1$  for some  $x' \in X$ .*

Next observe that Lemma 1, inequality (2), and our assumption on the cardinality of  $S$ , together provide the inequality

$$(\delta - 2)(g - 3) + \delta - 1 \geq |S| \geq |S_X| + |S_X^*| \geq |S_X| + (\delta - 1)(g - 2) - m'$$

which implies

$$(3) \quad m' \geq g - 3 + |S_X|.$$

However, since every edge in  $H$  corresponds to an edge or a path of length 2 in  $G$ , the girth of  $H$  is at least  $g/2$ . Hence,

$$m' \leq \text{ex}(g - 2, \{C_3, \dots, C_{\lfloor (g-1)/2 \rfloor}\}).$$

Since  $g - 2 \leq 2\lfloor (g - 1)/2 \rfloor$ , it follows by Theorem 1, that  $m' \leq g$ . Thus,  $m' \in \{g - 2, g - 1, g\}$ . That is, we have three separate cases to consider. We handle these cases next.

**Case 1.:** If  $m' = g - 2$  then  $|S_X| = 1$ , hence  $|X \setminus S_X| = g - 3$  and by Lemma 2 at least  $g - 3$  edges of  $H$  correspond to edges of  $G$ . Consequently, a cycle of length  $k$  in  $H$  leads to a cycle of length at most  $k + 1$  in  $G$ . Therefore,  $H$  does not contain any cycle, hence  $m' \leq g - 3$ , which is the required contradiction.

**Case 2.:** If  $m' = g - 1$  then  $|S_X| \leq 2$ , hence  $|X \setminus S_X| \geq g - 4$  and by Lemma 2 at least  $g - 4$  edges of  $H$  correspond to edges of  $G$ . Consequently, a cycle of length  $k$  in  $H$  leads to a cycle of length at most  $k + 3$  in  $G$ , and therefore the girth of  $H$  is at least  $g - 3$ . By Corollary 2 this implies  $m' \leq g - 2$ , which is the required contradiction.

**Case 3.:** If  $m' = g$  then  $|S_X| \leq 3$ , hence  $|X \setminus S_X| \geq g - 5$  and by Lemma 2 at least  $g - 5$  edges of  $H$  correspond to edges of  $G$ . Consequently, a cycle of length  $k$  in  $H$  leads to a cycle of length at most  $k + 5$  in  $G$ , and therefore the girth of  $H$  is at least  $g - 5$ . By Corollary 1 this implies  $m' \leq g - 1$  which is the required contradiction.

This completes the proof of the theorem, and the conjecture presented in [18] is resolved in the affirmative.  $\square$

### 3. CONCLUDING REMARKS

Let  $f(g, \delta)$  denote the minimum zero forcing number over all graphs of girth  $g$  and minimum degree  $\delta$ . Theorem 2 provides a lower bound for  $f$ , and from [18] we know that this bound is tight in the following cases:

- $f(g, 2) = 2$  for all  $g \geq 3$  (the  $g$ -cycle),
- $f(3, \delta) = \delta$  for all  $\delta \geq 1$  (the complete graph  $K_{\delta+1}$ ),
- $f(4, \delta) = 2\delta - 2$  for all  $\delta \geq 2$  (the complete bipartite graph  $K_{\delta, \delta}$ ),
- $f(4, 3) = 4$  (the 3-cube),
- $f(5, 3) = 5$  (the Petersen graph),
- $f(6, 3) = 6$  (the Heawood graph).

Consequently, the smallest open cases are the following.

**Question 1.** We know  $7 \leq f(7, 3) \leq 8$  and  $8 \leq f(8, 3) \leq 10$ . Can we close these gaps?

**Question 2.** We know  $f(5, 4) \geq 8$ . What is the best upper bound we can come up with?

Using essentially the same argument as in the proof of Theorem 2, one can prove that the bound (1) is not sharp in general (for instance  $f(g, \delta) \geq \delta + (g - 3)(\delta - 2) + 1$  for  $g \geq 14$ ,  $\delta \geq 3$ ). This motivates the following questions.

**Question 3.** What are upper bounds for  $f(g, \delta)$ ?

**Question 4.** What can be said about the asymptotic behavior of  $f(g, \delta)$ ?

In a slightly different direction, we can ask for better lower bounds on the zero forcing number under additional assumptions on the structure of the graph. For instance, it is known [22] that  $F(G) \geq F(G - e) - 1$  for every edge  $e$  of  $G$ . If  $G$  contains a bridge  $e$  whose removal leads to two connected components  $G_1$  and  $G_2$ , then this implies  $F(G) \geq F(G_1) + F(G_2) - 1$ , and if  $G$  has girth  $g$  and minimum degree  $\delta$  then

$$F(G) \geq 2[(\delta - 1) + (g - 3)(\delta - 3)] - 1,$$

which is a stronger bound than (1) whenever  $\delta \geq 4$ . For graphs of large girth the bound might be strengthened by combining the observation that every graph  $G$  can be partitioned into  $F(G)$  induced paths [5] with the Moore bound [3]: if  $G$  has girth  $g$ , average degree  $d$  and no induced path of length more than  $\ell$  then

$$F(G) \geq \begin{cases} \frac{d((d-1)^r-1)}{\ell(d-2)} & \text{if } g = 2r + 1 \\ \frac{2((d-1)^r-1)}{\ell(d-2)} & \text{if } g = 2r. \end{cases}$$

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